The Language of Mathematical Exposition
(Gradually Introduced Grades 7-12)

All teachers are concerned with vocabulary, sentence structure, and reading and writing skills. Mathematics teachers also employ a special symbolic language in which letters are used to represent statements, variables, etc. Essential components of this language are:

1. Set Language: Intersection $\cap$, union $\cup$, inclusions $\subseteq$, proper inclusion $\subset$, is a member of $\in$, is not a member of $\notin$, and empty set $\emptyset$. Set builder notation $\{x|\sqrt{x-3} > 5\}$.
2. Definitions and assumptions as a basis of reasoning.
3. Quantification involving "some", "all" ($\forall$) and "there exists" ($\exists$), etc. Single exception destroys an "all statement".
4. Statements. A statement is a positive declaration with no attempt at proof. Example: In any group of eight people at least two have birthdays on the same day of the week.
5. Equality. The statement "$a = b$" means that $a$ and $b$ are names for the same thing.
6. Conjuction. The statement "$p$ and $q$", written "$p \wedge q$" is the conjunction of $p$ and $q$. It says that both $p$ and $q$ are true.
7. Disjunction. The statement "$p$ or $q$", written "$p \vee q$" is the disjunction of $p$ and $q$. It says that at least one of the two statements is true.
8. Implication: The statement "$p$ implies $q$", written "$p \Rightarrow q$" is an implication. It can also be written as "If $p$ then $q$", where $p$ is the hypothesis and $q$ is the conclusion. An implication is true in all cases except when the hypothesis is true and the conclusion is false.
9. Equivalence. Two statements, $a$ and $b$, are logically equivalent, written "$a \Leftrightarrow b$", if each implies the other.
10. Negation. The statement "$p$ is false", written "$\sim p$", and read "not $p$"; is the negation of statement $p$.
11. Contradiction. A statement that is equivalent to the negation of $p$ is a contradiction of $p$ and can also be written as $\sim p$. Two statements are contradictory if exactly one of them is true; i. e. each is equivalent to the negation of the other. Example: *Some Scandinavians are vegetarians* $(S \cap V \neq \emptyset)$ and *No Scandinavians are vegetarians* $(S \cap V = \emptyset)$ are contradictory statements.
12. Double negative. $\sim(\sim p) \Leftrightarrow p$.
13. Contradicting conjunctions, disjunctions and implications.
    $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$
    $\sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q)$
    $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q) \quad \therefore \sim(p \Rightarrow q) \Leftrightarrow (p \wedge \sim q)$
14. Converse, inverse and contrapositive. For any implication $p \Rightarrow q$ we have
    Converse: $q \Rightarrow p$
    Inverse: $\sim p \Rightarrow \sim q$
    Contrapositive: $\sim q \Rightarrow \sim p$
15. Equivalence of an implication and its contrapositive:
    $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q) \Leftrightarrow [\sim p \vee \sim(\sim q)] \Leftrightarrow [\sim(\sim q) \vee \sim p] \Leftrightarrow (\sim q \Rightarrow \sim p)$

16. Distributive Laws: $[a \vee (b \wedge c)] \Leftrightarrow [(a \vee b) \wedge (a \vee c)]$;
$[a \wedge (b \vee c)] \Leftrightarrow [(a \wedge b) \vee (a \wedge c)]$.

17. Tautologies: A composite statement is a statement formed from other statements $a, b, c, \ldots$ by using some of the connectives $\wedge$, $\vee$ and $\Rightarrow$. The statements $a, b, c, \ldots$ are the components of the composite statement. A composite statement that is true for all possible truth values (T or F) of its components is called a tautology. Examples:

(i) $a \vee \sim a$ (ii) $\left.\begin{array}{c} p \\ (p \Rightarrow q) \end{array}\right\} \Rightarrow q$ or $[p \wedge (p \Rightarrow q)] \Rightarrow q$ (iii) $\left.\begin{array}{c} (\sim p \Rightarrow q) \\ \sim q \end{array}\right\} \Rightarrow p$

(iv) $[p \wedge q \wedge (p \Rightarrow x) \wedge (q \Rightarrow y) \wedge [(x \wedge y) \Rightarrow z]] \Rightarrow z$.

**Note:** (ii), (iii) and (iv) are proof patterns.

18. a. Partial contrapositive and partial converse. In most theorems of the form $p \Rightarrow q$, the hypothesis $p$ is the conjunction of several component statements. If there are four such component statements, the theorem takes this form:

$\left.\begin{array}{c} a \\ b \\ c \\ d \end{array}\right\} \Rightarrow q$ This statement has four partial contrapositives, each obtained by exchanging a contradiction of $q$ with a contradiction of one of the four statements in the hypothesis. **Each one of these contrapositive forms is equivalent to the original theorem**. (Some of these contrapositives may also have the same verbal sense.) This theorem also has four converses, each obtained by exchanging $q$ with one of the four statements in the hypothesis. **Each of these converses is a conjecture which may or may not be true.** (Some of these converses may be equivalent.)

b. Generalized definition of converses (Lazar) If a theorem has $m$ statements in it hypothesis and $n$ statments in its conclusion, a converse may be formed by exchanging any number of statements in the conclusion with the same number of statements in the hypothesis. Each of the $C_n^{m+n} - 1$ converses is a conjecture whose truth value must be determined, and some may have the same verbal expression.

19. An argument is an assertion that the conjunction of premises $a, b, c, \ldots$ implies a conclusion $x$. In bracket form it looks like this:

[I.] $\left.\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\right\} \Rightarrow x$. An implication like [I.] can be a tautology (Compare with 17 (iv)). If so, it is a valid argument.

20. Proof: A valid argument with true premises is a proof of its conclusion.