Lee County Invitational — January 26, 2008

Pre-Calculus Individual — Dunbar High School

For each of the following questions mark the best answer on your scantron. If the correct answer is not present, choose choice E for "None of the above."

1. What is the range of the function y=4cos(2x + 7) – 3

   A. [-4,4] B. [-4,-2] C. [-7,1] D. [3,11] E. NOTA

2. A circle with circumference 6π inches is the great circle for a sphere. Half the volume of the sphere is the volume of a right, circular cone with radius of 7inches. What is the height of the cone, in inches?

   A. $\frac{54}{49}$ B. $\frac{18}{49}$ C. $\frac{72}{49}$ D. $\frac{1728}{49}$ E.NOTA

3. If $\csc(x) = \frac{11}{5}$ for an angle x in quadrant II, what is cot(x)?

   A. $\frac{5\sqrt{6}}{24}$ B. $\frac{4\sqrt{6}}{5}$ C. $\frac{-5\sqrt{6}}{24}$ D. $\frac{-4\sqrt{6}}{5}$ E. NOTA

4. Which of the following accurately describes the following statement: "x is at least 5 units away from 4."

   A. $|x-4|\geq 5$ B. $|x-5|\geq 4$ C. $|x-4|>5$ D. $|x-5|>4$ E.NOTA

5. If f(x) is a continuous, odd function, g(x) is a continuous, even function and h(x) is a continuous function that is neither even nor odd, then how many of the following must be even functions?

   $[h(x)]^2$ g(x)–f(x) g(f(x)) h(g(x)) cos(f(x)) sin(g(x)) f(x)g(x)h(x)

   A. 2 B. 3 C. 4 D. 5 E. NOTA

6. What is the tangent of the acute angle formed by the intersection of the vectors (2,5,-1) and (-1,3,3)?

   A. $\frac{\sqrt{570}}{57}$ B. $\frac{\sqrt{570}}{10}$ C. $\frac{\sqrt{470}}{10}$ D. $\frac{\sqrt{470}}{47}$ E. NOTA

7. What is the sum of the real solutions of the following equation: $9^x + 3^{(x+1)} - 18 = 0$?

A. 3 B. 1 C. -2 D. –3 E. NOTA

8. If you are dealt five cards from a standard deck of 52 cards, what is the probability of being dealt a four of a kind, given that one of the cards dealt to you is a four?

A. $\frac{624}{{}_{52}C_5}$ B. $\frac{1248}{{}_{52}C_5}$ C. $\frac{2496}{{}_{52}C_5}$ D. $\frac{4992}{{}_{52}C_5}$ E.NOTA

9. This Scottish mathematician, born in 1550, is best known for the creation of logarithms whose discussion first appeared in *Mirifii logarithmorum canonis descriptio* in 1614. Who is he?

A. Leonhard Euler B. Carl Gauss C. John Napier
D. Fred Logarithm E. NOTA

10. A set is said to be *closed* under a certain operation if, when you take any two elements from the set and perform the operation on them, the resulting answer is also in the set. For example, the set of integers under addition is closed because the sum of any two integers is an integer. Which of the following sets are also closed under the given operation?

A. Irrational Numbers, Multiplication B. Natural Numbers, Subtraction
C. Integers, Division D. Real Numbers, Division E. NOTA

11. Solve for x over the real numbers: $2\tan(\frac{x}{2}) - 1 = 1$

A. $\pi + \pi \bullet k$, $k \in$ {integers} B. $\pi + 2\pi \bullet k$, $k \in$ {integers}
C. $\frac{\pi}{2} + \pi \bullet k$, $k \in$ {integers} D. $\frac{\pi}{2} + 2\pi \bullet k$, $k \in$ {integers} E. NOTA

12. Which of the following is/are true:

I. $i^{247} = -i$ where $i = \sqrt{-1}$
II. $[\text{cis}(\theta)]^n = \text{cis}(\theta^n)$ where $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$ and $i = \sqrt{-1}$
III. 17 is a complex number.

A. I only B. II only C. I and III only D. II and III only E. NOTA

13. An equation of the form $x^3 + 3x^2 + Ax - 24 = y$ has three, distinct integral roots that form an arithmetic sequence. What is A?

A. 22 B. –22 C. 10 D. –10 E. NOTA

14. Evaluate: $\sin(2\text{Sec}^{-1}(\frac{7}{3}))$

A. $\frac{12\sqrt{10}}{49}$ B. $\frac{4\sqrt{10}}{3}$ C. $\frac{-31}{7}$ D. $\frac{31}{7}$ E. NOTA

15. For the circle defined by the equation $x^2 + y^2 - 10x + 2y - 6 = 0$, what is the slope of the line tangent to the circle at the point (1,3)?

A. $\frac{1}{2}$ B. $\frac{-1}{2}$ C. 1 D. –1 E. NOTA

16. Mike is on "Press Your Luck." After the question round he heads to the "Big Board" with only 1 spin. However, Mike has done his research and he knows that on any spin of the big board he can pick up a bonus spin with probability of $\frac{1}{5}$. What is the total expected number of spins Mike will have at the Big Board?

A. $\frac{6}{5}$ B. $\frac{15}{7}$ C. $\frac{5}{4}$ D. $\frac{25}{16}$ E. NOTA

17. Solve for x: $\sin(x) - \sin^2(x) + \sin^3(x) - \sin^4(x) + \ldots = \frac{1}{2}$ where sin(x) does not equal 1 or –1.

A. $\frac{\pi}{2}$ B. $\frac{-\pi}{2}$ C. $\frac{\pi}{4}$ D. $\frac{-\pi}{4}$ E. NOTA

18. If $\cos(4x) = \frac{3}{8}$ and x is a Quadrant I angle, then what is $[\sin(2x)]^2$?

A. $\frac{5}{16}$ B. $\frac{495}{1024}$ C. $\frac{\sqrt{55}}{8}$ D. $\frac{55}{64}$ E. NOTA

Questions 19 and 20 refer to the following equation: $9x^2 - 4y^2 - 72x - 48y - 36 = 0$

19. What are the coordinates of the center of the hyperbola?

A. (-2, 3) B. (4, -6) C. (2,-3) D. (-4, 6) E. NOTA

20. What are the equations of the asymptotes of the hyperbola?

A. $y = \frac{2}{3}x - 12$, $y = \frac{-2}{3}x$ B. $y = \frac{-2}{3}x + 6$, $y = \frac{2}{3}x$ C. $y = \frac{-3}{2}x + 6$, $y = \frac{3}{2}x$ D. $y = \frac{3}{2}x - 12$, $y = \frac{-3}{2}x$ E. NOTA

21. In triangle ABC, $m\angle C = 30°$, c=2, $\frac{\sin(A)}{\sin(B)} = \sqrt{3}$ and the area of the triangle is $\sqrt{3}$. What is the perimeter of this triangle? (NOTE: Capital letters represent angles and lowercase letters represent side lengths)

A. $2 + \sqrt{3}$ B. $8\sqrt{3}$ C. $4 + 2\sqrt{3}$ D. $2 + 2\sqrt{3}$ E. NOTA

22. According to the Rational Root Theorem, what are all possible rational roots of the equation defined by $y = 100x^{100} + 99x^{99} + \ldots + 2x^2 + x + 1$?

A. 1, -1 B. 100, -100 C. $\frac{1}{100}, \frac{-1}{100}$ D. 0 E. NOTA

23. Simplify the following expression for natural numbers n>3: $\frac{(n+2)!({}_nP_4)}{({}_nC_4)({}_nP_{(n-1)})}$

A. $4n^2 + 12n + 8$ B. $n^2 + 3n + 2$ C. $(n + 1)!(n^2 + 3n + 2)$ D. $24n^2 + 72n + 48$ E. NOTA

24. For two non-empty sets A and B, simplify the following: $(A \cap B)' \cap (B \cup A')$ (NOTE: A' denotes the complement of set A)

A. B B. $A \cap B'$ C. A' D. $A' \cup B'$ E. NOTA

25. Where $\lceil x \rceil$ is the greatest integer function of x, evaluate the following:

$$\lceil \text{Cos}^{-1}(-1) \rceil + \lceil -e \rceil + \lceil 1.4 \rceil + \left\lceil \log \frac{1}{2000} \right\rceil$$

A. -3 B. -2 C. -1 D. 0 E. NOTA

26. Find the volume of the solid which results when the region bounded by the graphs y = 0, x = -3, x = 3 and y = |x| is revolved about the y-axis.

A. 9π B. 18π C. 27π D. 36π E. NOTA

27. Spurred by his success as the host of "Survivor", Jeff Probst opens a wilderness store specializing in exotic merchandise. One of Jeff's favorite items is his collection of "Survivor Tiki Torches" which he produces himself! Jeff has noticed that when the price of his torches is $10 he sells 400 of them. When he raises the price of the torches by $2 his sales fall by 40 torches. The relationship between the torches sold, x, and price of the torch, p, is linear. What is the equation expressing p as a function of x?

A. $p = \frac{-1}{20}x + 30$ B. p = -20x + 10010 C. $p = \frac{1}{20}x + 30$
D. p = 20x + 10010 E. NOTA

28. Given the function y = f(x), which of the following represents the graph of translating f(x) to the right 1 unit and then reflecting it about the line y = x?

A. $y = f^{-1}(x-1)$ B. $y = f^{-1}(x) + 1$ C. $y = f^{-1}(x+1)$ D. $y = f^{-1}(x) - 1$ E. NOTA

29. Solve the following equation for x: $12x^2 + 14x - 28y - 24xy = 0$

A. $\{\frac{7}{6}, 2y\}$ B. $\{\frac{-7}{6}, -2y\}$ C. $\{\frac{7}{6}, -2y\}$ D. $\{\frac{-7}{6}, 2y\}$ E. NOTA

30. Assume the following information: All hippogriffs are thestrals
All thestrals are grindylows
Some grindylows are pygmy puffs
No pygmy puffs are hippogriffs

Which of the following are invalid conclusions?

I. All hippogriffs are grindylows
II. No thestrals are pygmy puffs
III. There is a pygmy puff that is both a thestral and a grindylow

A. I and II B. II and III C. II only D. I, II and III E. NOTA

Given $f(2x + 1) = 3x^2 + 7x - 8$
$g(x - 5) = 2x^3 - 4x^2 + 10$
Evaluate:

A. $f(7)$
B. $g(-9)$
C. $g(f(1))$
D. $f(x)$

In a recent survey of 50 Girl Scout Cookie Enthusiasts, it was found that 20 eat Tagalongs, 25 eat Do-Si-Dos, 24 eat Samoas, 10 eat Tagalongs and Do-Si-Dos, 9 eat Do-Si-Dos and Samoas, 8 eat Tagalongs and Samoas and 3 eat none of Tagalongs, Do-Si-Dos and Samoas.

A. How many people eat Tagalongs, Do-Si-Dos and Samoas?
B. What is the probability that a person eats Tagalongs and Do-Si-Dos but not Samoas?
C. What is the probability a person eats only Do-Si-Dos?
D. What is the probability that a person eats Tagalongs given that they eat Samoas?

Lee County Invitational January 26, 2008
Precalculus Team Round 3

Given $\vec{u} = 3\vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{v} = 6\vec{i} - \vec{j}$

$\vec{w} = -2\vec{i} + 3\vec{j} + 5\vec{k}$

Determine the following:

A. A vector of length 7 in the direction of $\vec{v}$.

B. $\|3\vec{v} - 2\vec{w}\|$

C. $\vec{u} \times \vec{w}$

D. $(\vec{u} \times \vec{w}) \bullet \vec{v}$

Lee County Invitational January 26, 2008
Precalculus Team Round 4

A. What is the coefficient of the x^2y^3 term in the expansion of $(x - 2y)^5$?

B. What is the sum of the coefficients of the terms containing only x or y terms in the expansion of $(4w + 6x - 2y + z)^4$?

C. Solve for n: ${}_{n+2}C_n + {}_{n+1}C_{n-1} = 25$ for $n \in \mathbf{N}$

D. Let $f(m) = \sum_{n=0}^{m} {}_mC_n$ for $m \in \mathbf{N}$. What is f(1) + f(2) +…+ f(7)?

For the following questions let $i=\sqrt{-1}$ and let cis(x) = cos(x) + *i*sin(x).

A. What is the sum of the complex fifth roots of unity?
B. What is the sum $|\text{cis}(0°)|+|\text{cis}(1°)|+...+|\text{cis}(90°)|$?
C. Evaluate the following quotient in rectangular form: $\frac{(2\text{cis}(30°))^4}{(3\text{cis}(15°))^2}$
D. Evaluate the following infinite sum in rectangular form: $e^{\frac{\pi}{6}i}+\frac{1}{2}e^{\frac{7\pi}{6}i}+\frac{1}{4}e^{\frac{13\pi}{6}i}+...$

Given that $\Delta ABC \sim \Delta DEF$, BC = 4, DF = $\frac{\sqrt{6}}{2}$, $m\angle B=60°$ and $m\angle D=45°$ determine the following:

A. What is the ratio of similarity, in fractional form, of $\Delta DEF:\Delta ABC$?
B. What fraction of the area of ΔABC is the area of ΔDEF?
C. What is the length of DE?
D. What is the length of the median to side B in ΔABC?

Determine A-D given the following matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 7 & -1 \\ 0 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 6 & -2 \\ 3 & 3 & -4 \end{pmatrix} \quad C = \begin{pmatrix} 10 & -2 \\ -5 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & 6 \\ 7 & 0 & -2 \end{pmatrix} \quad E = \begin{pmatrix} 5 & 7 & 2 \\ 0 & -1 & -4 \\ 1 & 2 & 6 \end{pmatrix} \quad F = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

A. What is the sum of the entries in AB?
B. How many of the above matrices have inverses?
C. What is the determinant of D?
D. If a matrix is chosen, replaced and then a second matrix (not necessarily distinct) is chosen, what is probability that the operation of the first matrix times the second matrix (in that order) can be performed?

Given the following parametric equations:

$$x = \sqrt{2} + 2\sin(t)$$
$$y = 2\cos(t)$$

A. Determine the value of x in the above equations if $y = \sqrt{3}$ and $\frac{3\pi}{2} \le t \le 2\pi$.
B. If $0 \le t \le \frac{\pi}{4}$ determine the value of t when the graph characterized by the above equations intersects the line y = x.
C. Which of the following best describes the graph defined by the above parametric equations: hyperbola, ellipse, circle, parabola, one line, two intersecting lines, a point?
D. What is the area enclosed by the graph characterized by the above equations (if the graph has no enclosed area write “none”)?

If the points (-1,1), (2,2), (6,0) lie on a circle, determine the following:

A. The radius of the circle determined by the above points.
B. The center of the circle determined by the above points.
C. The shortest distance from (10,1) to the circle determined by the above points.
D. The area within the circle defined by the above points that lies below the line x-2y=8.

Use the fact that sinh(x) = $\frac{e^x - e^{-x}}{2}$ and cosh(x) = $\frac{e^x + e^{-x}}{2}$ to answer the following questions:

A. How many times does sinh(x) = cosh(x)?
B. Evaluate tanh(2ln4).
C. Solve for x where coth(x) = 2.
D. $\sinh^2(x) + \cosh^2(x)$ can be written in the form Acosh(Bx) where A,B $\in$ **Z**. What is A+B?

Lee County Invitational January 26, 2008
Precalculus Team Round 11

A. Solve for A if the distance between the rectangular coordinates (3,A) and (5,-2) is 6 and A>0.
B. Determine the distance between the polar coordinates (5,0) and $(-2, \frac{4\pi}{3})$.
C. Determine the distance between the spherical coordinates $(2, \frac{\pi}{3}, \frac{\pi}{2})$ and $(-1, 0, \frac{\pi}{6})$.
D. Determine the shortest distance between the point (2,-4) and the line 6x – 7y = 3.

Lee County Invitational January 26, 2008
Precalculus Team Round 12

Given the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ determine the exact values of the following summations:

A. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$
B. $\sum_{n=1}^{\infty} \frac{2^n + n^2}{2^n n^2}$
C. $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$
D. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Lee County Invitational — January 26, 2008
Precalculus Team — Round 13

Use the following equation for answering parts A-D: $f(x) = x^5 – 7x^4 + 16x^3 – 38x^2 + 48x – 40$

A. Using Descartes' Rule of Signs, what is the maximum number of positive roots for f(x)?
B. Using Descartes' Rule of Signs, what is the maximum number of negative roots for f(x)?
C. What is the maximum number of roots of f(x) that can be written in the form $x = a + bi$ where $i=\sqrt{-1}$ and $b \neq 0$?
D. If two of the solutions of f(x) are $2i$ and $1-i$ where $i=\sqrt{-1}$, what is the sum of the other three roots?

Southwest Florida Invitational — January 26, 2008
Precalculus Team — Round 14

For each of the following statements determine if they are true or false. Please write either "TRUE" or "FALSE."

A. $1 + \tan^2 x = \sec^2 x$ for all $x \in \mathbf{R}$.
B. The inverse of the function $f(x) = \log(2e^x + 7)$ is $f^{-1}(x) = \ln(\frac{10^x - 7}{2})$.
C. The solution set to the inequality $\frac{2x - 3}{3x - 8} \geq 1$ is $[\frac{8}{3}, 5]$.
D. If 2+2=3, then the Earth is flat.

A. What is the greatest common divisor of 28 and 74?
B. What is the least common multiple of 28 and 74?
C. What is the greatest common divisor of 156 and 384?
D. What is the least common multiple of 156 and 384?

January 2008 Southwest Florida Invitational
Precalculus Individual Answers

1. C
2. A
3. D
4. A
5. C
6. C
7. B
8. B
9. C
10. E
11. D
12. C
13. B
14. A
15. C
16. C
17. E
18. A
19. B
20. D
21. C
22. E
23. D
24. C
25. A
26. B
27. A
28. B
29. D
30. B

January 2008 Southwest Florida Invitational
Precalculus Individual Solutions

1. The range of the cosine function is normally [-1,1]. When multiplied by 4 (the amplitude), the range becomes [-4,4]. Finally, when subtracting 3 from the entire function (the vertical shift) it moves all points in the range down 3; hence [-7,1] **C.**
   $2\pi r = 6\pi \Rightarrow r = 3$
2. $V = \frac{4}{3}\pi r^3 \Rightarrow V = \frac{4}{3}\pi(3)^3 \Rightarrow V = 36\pi$
   $18\pi = \frac{1}{3}\pi(7)^2 h \Rightarrow h = \frac{54}{49}$ **A**
3. Draw the triangle in quadrant II and get a triangle with side lengths of 5, $-4\sqrt{6}$ (along the negative x-axis) and 11. So cot(x) = $\frac{\text{adjacent}}{\text{opposite}} = \frac{-4\sqrt{6}}{5}$ **D.**
4. **A.** by definition
5. $[h(-x)]^2 \neq [h(x)]^2$ so NOT even
   g(-x) – f(-x) = g(x) – [-f(x)] = g(x)+f(x) $\neq$ g(x)-f(x) so NOT even
   g(f(-x)) = g(-f(x)) = g(f(x)) so EVEN
   h(g(-x)) = h(g(x)) so EVEN
   cos(f(-x)) = cos(-f(x)) = cos(f(x)) so EVEN
   sin(g(-x)) = sin(g(x)) so EVEN
   f(-x)g(-x)h(-x) $\neq$ f(x)g(x)h(x) so NOT even
   Therefore **C.**
6. 2(-1) + 5(3) + (-1)(3) = $\sqrt{2^2+5^2+(-1)^2}\sqrt{(-1)^2+3^2+3^2}\cos\theta \Rightarrow \frac{10}{\sqrt{570}} = \cos\theta$
   Drawing a triangle in quadrant I gives that $\tan\theta = \frac{\sqrt{470}}{10}$ **C.**
7. $9^x + 3^{(x+1)} - 18 = 0 \Rightarrow (3^x)^2 + 3(3^x) - 18 = 0$
   Substitute $u = 3^x$
   So $u^2 + 3u - 18 = 0$ Thus u=3,-6
   Then $3=3^x$ so x=1
   $-6=3^x$ so no real solution
   Thus the sum of the real solutions is 1**B.**
8. P(4 of a kind|1 four) = P(4-4's|1 four) + P(4 of a kind of anything but 4s|1 four) =
   $$\frac{\frac{(4C4)(48C1)+(4C1)(12(4C4))}{52C5}}{\frac{1}{13}} = \frac{1248}{52C5}$$ **B.**
9. C. John Napier

10. Choice A is NOT closed since $\sqrt{2}\sqrt{2} = 2 \notin irrationals$. Choice B is NOT closed since 1-2 = -1 $\notin$ natural numbers. Choice C is NOT closed since $\frac{4}{8} = .5 \notin Integers$. Choice D is NOT closed since $\frac{4}{0}$ is undefined and thus not in the set of real numbers. Thus the answer is **E.**

11. Solving yields $\tan\frac{x}{2} = 1$. So $\frac{x}{2} = \tan^{-1}(1) = \frac{\pi}{4}$. So $x = \frac{\pi}{2}$. Now we must add on the period of the function which for the tangent function is $\frac{\pi}{\frac{1}{2}} = 2\pi$. So all solutions can be written as $\frac{\pi}{2} + 2\pi\mathbf{Z}$. Thus choice **D.**

12. I. $i^{247} = i^{244}i^3 = (i^4)^{61}i^3 = 1(-i) = -i$ TRUE
II. $[\text{cis}(\theta)]^n = \text{cis}(n\theta)$ FALSE
III. Since the real numbers are a subset of the complex numbers, 17 is a complex number. TRUE.
Thus I and III only **C.**

13. Let the 3 roots of the equation be a,b,c. Letting a be the first term of the arithmetic sequence means that b=a+r and c=a+2r where r is the common ratio of the sequence. Now by properties of functions, the sum of the roots is –3/1=-3. So a+b+c=a+(a+r)+(a+2r)=-3. So 3a+3r=-3. So a+r= -1. Also, the product of the roots is –(-24)/1=24. Notice the product is also written as abc=a(a+r)(a+2r), but a+r= -1 by above so the product can be written as a(-1)(a+r+r)=a(-1)(-1+r). Now since a+r= -1 so a=-1-r. So the product can now be written as $(-r-1)(-1)(r-1)=(r+1)(r-1)=r^2-1=24$ so r=5 (or r=-5, either solution produces the desired answer). Substituting back yields that a= -1-5=-6. So a= -6, b= -1 and c=4. Now that the roots have been found, one way to find A is to take the product 2 at a time: (-6)(-1) + (-6)(4) + (-1)(4) = -22. **B.**

14. Allow $\text{Sec}^{-1}(\frac{7}{3}) = \theta$. So $\text{Sec}\,\theta = \frac{7}{3}$. So $\sin(2\text{Sec}^{-1}(\frac{7}{3})) = \sin(2\theta) = 2\sin\theta\cos\theta$. Using triangles with the secant yields this to be equal to $2\frac{\sqrt{40}}{7}\frac{3}{7} = \frac{2\sqrt{10}}{49}$. Thus choice **A.**

15. Completing the square yields the circle to have the equation $(x-5)^2 + (y+1)^2 = 32$. Recall from geometry that tangents to circles are always perpendicular to the radius of the circle. So, by finding the slope of the radius of the circle from the center to the desired point, we will be able to find the slope of the tangent. The slope of the radius is $\frac{(-1)-3}{5-1} = \frac{-4}{4} = -1$. So the slope of the tangent is the opposite reciprocal, or 1, choice **C**.

16. E(number of spins) = 1(P(only 1 spin)) + 2(P(only 2 spins)) + 3(P(only 3 spins)) + 4(P(only 4 spins) + … = 1(4/5) + 2(1/5)(4/5) + 3(1/5)(1/5)(4/5) + 4(1/5)(1/5)(1/5)(4/5) + … =
$4[\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots] = 4[\frac{5}{(5-1)^2}] = 4[\frac{5}{16}] = \frac{5}{4}$ **C.**

17. By the infinite geometric series formula $\frac{\sin x}{1-(-\sin x)} = \frac{1}{2}$. So 2sinx = 1 + sinx. So sinx = 1. However, from the information in the questions sinx could not be 1 or -1, therefore there is no solution **E.**

18. $[\sin(2x)]^2$ = (1/2)[1 – cos(4x)] = (1/2)(1 – 3/8) = (1/2)(5/8) = 5/16 **A.**

19. By completing the square $9(x^2 - 8x + 16) - 4(y^2 + 12y + 36) = 36 + 9(16) - 4(36) \Rightarrow$ $\frac{(x-4)^2}{4} - \frac{(y+6)^2}{9} = 1$. Thus the center is at (4,-6) **B.**

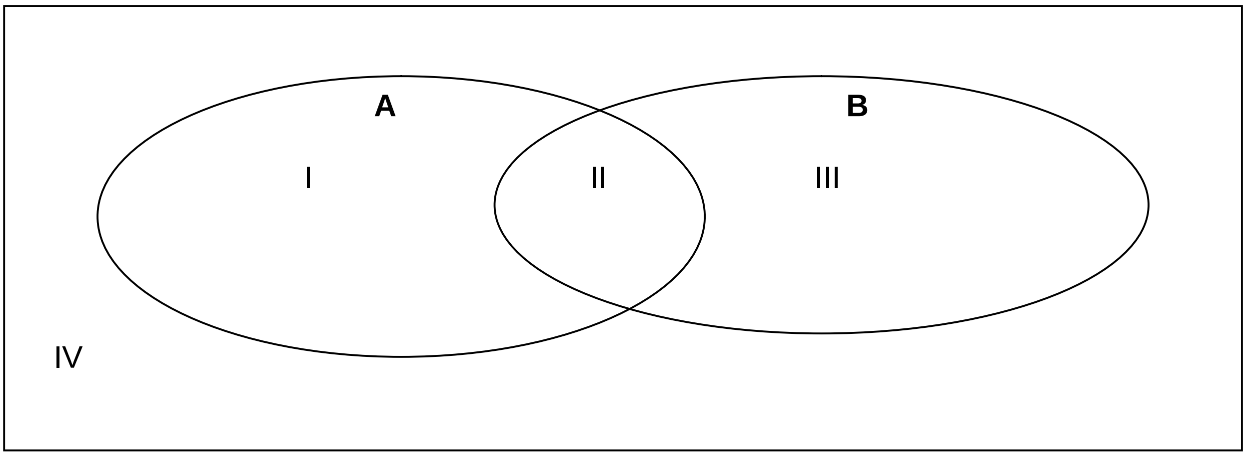
20. ANSWERS

21. First, by the Law of Sines $\frac{\sin(A)}{\sin(B)} = \frac{a}{b} = \sqrt{3}$. So $a = \sqrt{3}b$. Now using the area formula A = $\frac{1}{2}ab\sin C \Rightarrow \sqrt{3} = \frac{1}{2}(\sqrt{3}b)b\sin(30°) \Rightarrow \sqrt{3} = \frac{\sqrt{3}}{4}b^2 \Rightarrow b^2 = 4 \Rightarrow b = 2.$ Then $a = \sqrt{3}(2) = 2\sqrt{3}.$ Thus, the perimeter of the triangle is $4+2\sqrt{3}$ choice **C.**

22. By the Rational Root Theorem, all possible rational roots are constructed by taking factors of the constant and dividing them by the factors of the coefficient of the leading term. Since 50 is a factor of the leading term and 1 is a factor of the constant, 1/50 is a possible rational root. However, since no answer gives this choice the answer is **E.**

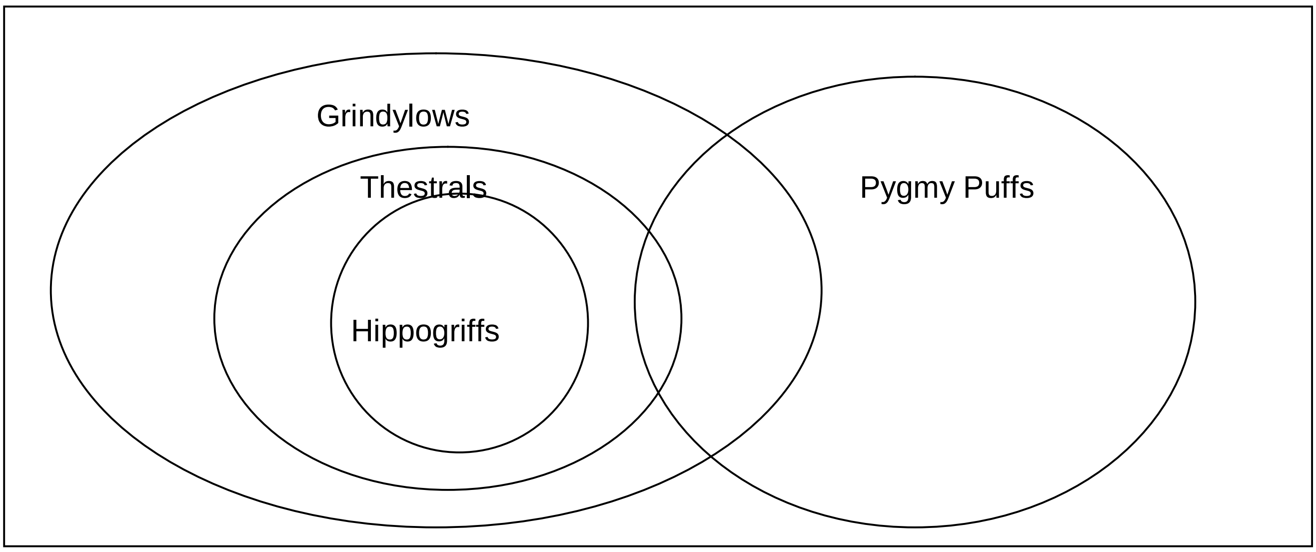
23. $$\frac{(n+2)!({}_nP_4)}{({}_nC_4)({}_nP_{n-1})} = \frac{(n+2)!\frac{n!}{(n-4)!}}{\frac{n!}{4!(n-4)!}\,\frac{n!}{(n-(n-1))!}}$$
$$= \frac{1!4!(n+2)!(n-4)!n!}{(n-4)!n!n!}$$
$$= \frac{24(n+2)(n+1)n!}{n!}$$
$$= 24(n^2+3n+2) = 24n^2+72n+48$$ **D.**

24.



First, $A \cap B$ contains only region II so $(A \cap B)'$ contains regions I, III and IV. Next, A' contains regions III and IV, and B contains regions II and III so $(B \cup A')$ contains regions II,III and IV. So finally $(A \cap B)' \cap (B \cup A')$ contains the intersection of the two regions which will be regions III and IV. This corresponds to the region occupied by A' **C.**

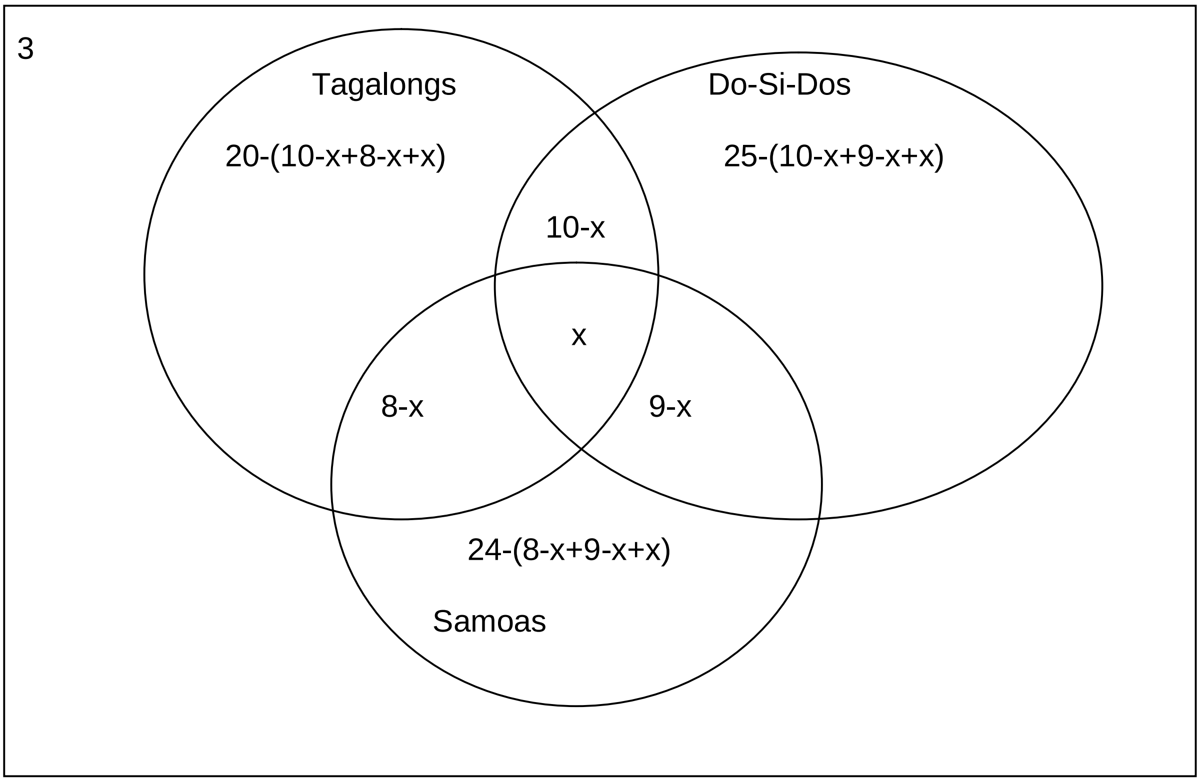
25. First, notice that $\log(1/2000) = \log(2000)^{-1} = -\log(2000)$. Also, log(2000) is greater than 3 and less than 4 since log(1000)=3 and log(10000)=4 and the logarithm function is continuous. So $\lceil \text{Cos}^{-1}(-1) \rceil + \lceil -e \rceil + \lceil 1.4 \rceil + \left\lceil \log \frac{1}{2000} \right\rceil = \lceil 3.14 \rceil + \lceil -2.718 \rceil + \lceil 1.4 \rceil + \lceil -3.\ldots \rceil = 3+(-3)+1+(-4) = -3$ **A.**
26. When the graph of |x| is revolved around the y-axis, the resultant shape is a cylinder with a cone taken out of the top. The cylinder has a base radius of 3 and a height of 3 so the volume is $V= \pi(3)^2(3) = 27\pi$. The cone also has a base radius of 3 and a height of 3 so the volume of the cone is $V = (1/3)\ \pi(3)^2(3) = 9\pi$. So the volume of the solid is $27\pi - 9\pi = 18\pi$ **B.**
27. We know that one point on the line is (400,10). Another point that can be found by the information given is (360,12). So the slope of the line is (10-12)/(400-360) = -1/20. So the equation of the line, by point-slope form, is (y-10) = (-1/20)(x-400). This reduces to y = (-1/20)x+30. **A.**
28. First, translating the line to the right one unit would transform the graph to y = f(x-1). Now, we must reflect this function across the line y=x, which means to take the inverse. So $x= f(y-1) \Rightarrow f^{-1}(x) = y-1 \Rightarrow f^{-1}(x) + 1 = y$ **B.**
29. Factoring by grouping 2x(6x+7) – 4y(7+6x) = 0. Then (2x-4y)(6x+7) = 0. Then x=2y,-7/6. **D.**
30. Venn Diagrams can be drawn to show that both II and III are false statements, however I is valid no matter how the diagram is drawn, for example, the following diagram leaves the hypotheses true but disproves II, To disprove III, move the Pygmy Puff circle to intersect only the Grindylow circle. Thus **B.**:



January 2008 Southwest Florida Invitational
Precalculus Team Round Solutions

1. A. For f(7), set x=3 in f(2x+1). So f(2(3)+1) = f(7) = 3(9) + 7(3) – 8 = 40
   B. g(-9) = g(-4 – 5) = 2(-64) – 4(16) +10 = -182
   C. f(1) = f(2(0) +1) = 3(0) + 7(0) -8 = -8
   g(-8) = g(-3-5) = 2(-27) – 4(9) +10 = -80
   D. f(x) = f(2(-1/2 + 1/2x) +1) = 3(-1/2 + 1/2x)^2 + 7(-1/2 +1/2x) – 8 = (3/4)x^2 +2x – (43/4)

2. Let x = the number of people who eat Tagalongs, Do-Si-Dos and Samoas.



Now (20-(10-x+8-x+x))+(25-(10-x+9-x+x))+(10-x)+x+(8-x)+(9-x)+(24-(8-x+9-x+x)) = 47
So x=5

A. 5
B. (10-5)/50 = 1/10
C. 11/50
D. P(Tagalongs|Samoas) = P(Tagalongs and Samoas)/P(Samoas) = (8/50)/(24/50) = 1/3

3. A. First, the unit vector of $\vec{v} = \frac{6}{\sqrt{37}}\vec{i} - \frac{1}{\sqrt{37}}\vec{j}$. Now multiplying this unit vector by 7 yields the answer: $\frac{42\sqrt{37}}{37}\vec{i} - \frac{7\sqrt{37}}{37}\vec{j}$

B. $3\vec{v} - 2\vec{w} = 22\vec{i} - 9\vec{j} - 10\vec{k}$. So $\left\|3\vec{v} - 2\vec{w}\right\| = \sqrt{22^2 + (-9)^2 + (-10)^2} = \sqrt{665}$

C. $\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & -2 \\ -2 & 3 & 5 \end{vmatrix} = 26\vec{i} - 11\vec{j} + 17\vec{k}.$

D. $(\vec{u} \times \vec{w}) \bullet \vec{v} = (26\vec{i} - 11\vec{j} + 17\vec{k}) \bullet (6\vec{i} - \vec{j}) = 26(6) + (-11)(-1) + 17(0) = 167$

4. A. $({}_5C_2)(1)^2(-2)^3 = 10(1)(-8) = -80$
B. Let w,z=0 and x,y=1. Then $(4(0) + 6(1) - 2(1) + 0)^4 = 4^4 = 256$

$$\binom{n+2}{n} + \binom{n+1}{n-1} = 25 \Rightarrow \frac{(n+2)!}{n!2!} + \frac{(n+1)!}{(n-1)!2!} = 25 \Rightarrow \frac{1}{2}(n+2)(n+1) + \frac{1}{2}(n+1)(n) = 25$$

C. $\Rightarrow (n-4)(n+6) = 0 \Rightarrow n = 4$

D. Notice $f(1) = {}_1C_0 + {}_1C_1 = 1+1 = 2^1$
$f(2) = {}_2C_0 + {}_2C_1 + {}_2C_2 = 1+2+1 = 2^2$
…..$f(7) = 2^7$

So $f(1) + f(2) + \ldots + f(7) = 2 + 2^2 + \ldots + 2^7 = 2\left(\frac{1-2^7}{1-2}\right) = 2(2^7 - 1) = 254$

5. A. This can be written as $x^5 = 1$ or $x^5 - 1 = 0$ where x are the complex fifth roots of unity. Then the sum of the roots is 0 by the formula for sum of the roots of polynomials.
B. Notice that the length of each of these will be 1 since cis(x) = cos(x) + isin(x) and the length is found by $\sqrt{\cos^2(x) + \sin^2(x)} = 1$ always. So the sum is 91 since there are 91 terms.

C. $\frac{(2\text{cis}(30°))^4}{(3\text{cis}(15°))^2} = \frac{16cis(120°)}{9cis(30°)} = \frac{16}{9}cis(90°) = \frac{16}{9}(\cos(90°) + i\sin(90°)) = \frac{16}{9}i$

D. Notice the sum is an infinite geometric sequence with first term $e^{\frac{\pi}{6}i}$ and common ratio $\frac{1}{2}e^{\pi i} = \frac{-1}{2}$. So the sum is $\frac{e^{\frac{\pi}{6}i}}{1 - \frac{-1}{2}} = \frac{\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})}{\frac{3}{2}} = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}i}{\frac{3}{2}} = \frac{\sqrt{3}}{3} + \frac{1}{3}i$

6. By similarity, we know that ∠E = 60° and ∠A = 45°. Now by Law of sines,
$\frac{\sin(60°)}{\frac{\sqrt{6}}{2}} = \frac{\sin(45°)}{EF} \Rightarrow EF = 1.$

A. Since BC~EF then the ratio of similarity is ¼.

B. Notice that DE = 1/4AB and EF=1/4BC. The area of ΔDEF =

$$\frac{1}{2}(\frac{1}{4}AB)(\frac{1}{4}BC)\sin(60°) = \frac{1}{16}(\frac{1}{2}AB \bullet BC\sin(60°)) = \frac{1}{16}Area\Delta ABC$$

C. Notice ∠F = 75°. So by the law of sines $\frac{\sin(45°)}{1} = \frac{\sin(75°)}{DE}$ Now using the sum formula for sine by writing sin(75) = sin(45+30) and rationalizing, we get that DE = $\frac{1+\sqrt{3}}{2}$

D. First, since we know that DE = $\frac{1+\sqrt{3}}{2}$ we can easily determine that AB = 2+2$\sqrt{3}$ by similarity and AC = $2\sqrt{6}$. Then using the formula for length of a median we get that the median length is $\sqrt{\frac{4^2}{2}+\frac{(2+2\sqrt{3})^2}{2}-\frac{(2\sqrt{6})^2}{4}} = \sqrt{10+4\sqrt{3}}$.

7. A. UV=$\begin{pmatrix} 2 & 3 \\ 7 & -1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 & -2 \\ 3 & 3 & -4 \end{pmatrix} = \begin{pmatrix} 11 & 21 & -16 \\ 4 & 39 & -10 \\ 15 & 15 & -20 \end{pmatrix}$. So the sum of the entries is 59.

B. Only square matrices that have a non-zero determinant are invertible. Only X and Y fit this, thus 2.

C. det(X) = 1((-1)(-2)-6(0))-2(3(-2)-6(7))+3(3(0)-(-1)(7)) = 119

D. Notice that there are 36 possibilities of choosing 2 matrices. If U is the first matrix only V or W can be the second for the multiplication to work. If V is first only U,X,Y or Z can be second. If W is first only V or W can be second. If X is first only U,X,Y or Z can be second. If Y is first only U,X,Y,Z can be second. If Z is first no matrix can be second. So the probability is 16/36=4/9

8. A. Substitute the value of y into the equation to find that t=$\frac{11\pi}{6}$. Substituting this value of t into the first equation to find x yields x=$\sqrt{2}$-1.

B. Solve $\sqrt{2}+2\sin(t) = 2\cos(t) \Rightarrow \sqrt{2} = 2(\cos(t)-\sin(t))$

Squaring both sides yields: $2 = 4(\cos^2(t)-2\sin(t)\cos(t)+\sin^2(t)) \Rightarrow 2 = 4(1-\sin(2t))$

So $-2 = -4\sin(2t) \Rightarrow \frac{1}{2} = \sin(2t) \Rightarrow \frac{\pi}{6} = 2t \Rightarrow \frac{\pi}{12} = t$

C. Notice $(x-\sqrt{2})^2 = 4\sin^2(t)$

$y^2 = 4\cos^2(t)$

So $(x-\sqrt{2})^2 + y^2 = 4(\sin^2(t)+\cos^2(t)) = 4$

This is the equation of a circle

D.The circle above has radius of 2, so the enclosed area is 4π.

9. Set up the following system: $(-1-h)^2 + (1-k)^2 = r^2$

$(2-h)^2 + (2-k)^2 = r^2$

$(6-h)^2 + (0-k)^2 = r^2$

A. Solving the system for r yields that the radius of the circle is 5
B. Solving the system for (h,k) yields that the center of the circle is (2,-3)
C. Notice the distance between the center of the circle and (10,1) is $\sqrt{(10-2)^2+(1+3)^2}=4\sqrt{5}$.
So subtracting off the radius gives us that the shortest distance to the circle is $4\sqrt{5}$-5.
D. Notice that the center of the circle falls on the line, thus the line is a radius and exactly half of the circle falls below the line. Thus half of the area of the circle is $\frac{25\pi}{2}$.

10. A. $\frac{e^x-e^{-x}}{2}=\frac{e^x+e^{-x}}{2}\Rightarrow e^x-e^{-x}=e^x+e^{-x}\Rightarrow 0=2e^{-x}\Rightarrow 0=\frac{2}{e^x}$. This equation has 0 solutions and thus sinh(x) and cosh(x) never intersect.
B. tanh(2ln4) = tanh(ln16) = $\frac{\sinh(\ln 16)}{\cosh(\ln 16)}=\frac{255}{257}$
C. coth(x) = $\frac{\cosh(x)}{\sinh(x)}=\frac{e^x+e^{-x}}{e^x-e^{-x}}=2\Rightarrow e^x+e^{-x}=2e^x-2e^{-x}$
So $3e^{-x}=e^x\Rightarrow 3=(e^x)^2\Rightarrow\sqrt{3}=e^x\Rightarrow\ln(\sqrt{3})=\frac{1}{2}\ln(3)=x$
D. $\sinh^2(x)+\cosh^2(x)=e^{2x}+e^{-2x}=2(\frac{e^{2x}+e^{-2x}}{2})=2\cosh(2x)$. So A+B=4

11 A. $\sqrt{(3-5)^2+(A+2)^2}=6\Rightarrow 4+(A+2)^2=36\Rightarrow(A+2)^2=32\Rightarrow A=4\sqrt{2}-2$
B. (5,0)$\Rightarrow x=5\cos 0=5; y=5\sin 0=0\Rightarrow(5,0)$
$(-2,\frac{4\pi}{3})\Rightarrow x=-2\cos(\frac{4\pi}{3})=1; y=-2\sin(\frac{4\pi}{3})=\sqrt{3}\Rightarrow(1,\sqrt{3})$
$\sqrt{(5-1)^2+(0-\sqrt{3})^2}=\sqrt{16+3}=\sqrt{19}$
C. $(2,\frac{\pi}{3},\frac{\pi}{2})\Rightarrow x=2\cos\frac{\pi}{3}\sin\frac{\pi}{2}=1; y=2\sin\frac{\pi}{3}\sin\frac{\pi}{2}=\sqrt{3}; z=2\cos\frac{\pi}{2}=0\Rightarrow(1,\sqrt{3},0)$
$(-1,0,\frac{\pi}{6})\Rightarrow x=-1\cos 0\sin\frac{\pi}{6}=\frac{-1}{2}; y=2\sin 0\sin\frac{\pi}{6}=0; z=-1\cos\frac{\pi}{6}=\frac{-\sqrt{3}}{2}\Rightarrow(\frac{-1}{2},0,\frac{-\sqrt{3}}{2})$
$\sqrt{(1+\frac{1}{2})^2+(\sqrt{3}-0)^2+(0+\frac{\sqrt{3}}{2})^2}=\sqrt{\frac{9}{4}+3+\frac{3}{4}}=\sqrt{6}$
D. 6x-7y-3=0, (2,-4)
$\frac{|6(2)+(-7)(-4)+(-3)|}{\sqrt{6^2+7^2}}=\frac{37}{\sqrt{85}}=\frac{37\sqrt{85}}{85}$

12. A. $\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}=\frac{1}{2^2}+\frac{1}{3^2}+...=(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...)-1=\sum_{n=1}^{\infty}\frac{1}{n^2}-1=\frac{\pi^2}{6}-1$

B. $\sum_{n=1}^{\infty}\frac{2^n+n^2}{2^n n^2}=\sum_{n=1}^{\infty}\frac{2^n}{2^n n^2}+\sum_{n=1}^{\infty}\frac{n^2}{2^n n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{\pi^2}{6}+\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\pi^2}{6}+1$

C. $\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\sum_{n=1}^{\infty}\frac{1}{4n^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{24}$

D. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...)=(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...)-(\frac{2}{2^2}+\frac{2}{4^2}+...)=\sum_{n=1}^{\infty}\frac{1}{n^2}-2\sum_{n=1}^{\infty}\frac{1}{(2\text{n})^2}$

$=\frac{\pi^2}{6}-2(\frac{\pi^2}{24})=\frac{\pi^2}{12}$

13. A. Since the coefficients of f(x) change sign 5 times, the max number of positive roots is 5.
B. f(-x) = $-x^5-7x^4-16x^3-38x^2-48x-40$. Since the coefficients of f(-x) never change sign, there are 0 negative roots.
C. Since f(x) is an odd function it must have at least 1 real root, therefore there are 4 imaginary roots possible.
D. The sum of the roots of f(x) is –(-7)/1=7. Then the sum of the other 3 roots can be found by 7-(2*i*+1-*i*)=6-*i*.

14. A. FALSE: tanx is not defined over all real numbers
B. TRUE
C. FALSE: 8/3 is not in the solution set
D. TRUE: false⇒false is true.

15. Notice: $28=2^2x7$
$74=2x37$

A. 2
B. LCM=2(2)(7)(37)=1036

Notice: $156=2^2x3x13$
$384=2^7x3$

C. $GCD=2^2x3=12$
D. LCM=2(2)(3)(2)(2)(2)(2)(2)(13)=4992

January 2008 Southwest Florida Invitational
Precalculus Team Answers

1. A. 40
   B. -182
   C. -80
   D. $\frac{3}{4}x^2 + 2x - \frac{43}{4}$

2. A. 5
   B. $\frac{1}{10}$
   C. $\frac{11}{50}$
   D. $\frac{1}{3}$

3. A. $\frac{42\sqrt{37}}{37}\vec{i} - \frac{7\sqrt{37}}{37}\vec{j}$
   B. $\sqrt{665}$
   C. $26\vec{i} - 11\vec{j} + 17\vec{k}$
   D. 167

4. A. -80
   B. 256
   C. 4
   D. 254

5. A. 0
   B. 91
   C. $\frac{16i}{9} or \frac{16}{9}i$
   D. $(\frac{\sqrt{3}}{3} + \frac{1}{3}i) or \frac{\sqrt{3}+i}{3}$

6. A. $\frac{1}{4}$
   B. $\frac{1}{16}$
   C. $\frac{1+\sqrt{3}}{2}$
   D. $\sqrt{10+4\sqrt{3}}$

7. A. 59
   B. 2
   C. 119
   D. $\frac{4}{9}$

8. A. $\sqrt{2}-1$
   B. $\frac{\pi}{12}$
   C. circle
   D. 4π

9. A. 5
   B. (2,-3)
   C. $4\sqrt{5}-5$
   D. $\frac{25\pi}{2}$

10. A. 0
    B. $\frac{255}{257}$
    C. $\frac{1}{2}\ln(3) \, or \, \ln(\sqrt{3})$
    D. 4

11. A. $\sqrt{32}-2$
    B. $\sqrt{19}$
    C. $\sqrt{6}$
    D. $\frac{37\sqrt{85}}{85}$

12. A. $(\frac{\pi^2}{6} - 1) or \frac{\pi^2 - 6}{6}$
    B. $(\frac{\pi^2}{6} + 1) or \frac{\pi^2 + 6}{6}$
    C. $\frac{\pi^2}{24}$
    D. $\frac{\pi^2}{12}$

13. A. 5
    B. 0
    C. 4
    D. 6-*i*

14. A. FALSE
    B. TRUE
    C. FALSE
    D. TRUE

15. A. 2
    B. 1036
    C. 12
    D. 4992